Antiphase synchronization of two nonidentical pendulums

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We numerically study the synchronization of two nonidentical pendulum motions, pivoting on a common movable frame in the point of view of the dynamic phase transition. When the difference in the pendulum lengths is not too large, it is shown that the system settles down into the dynamic state of the antiphase synchronization with the phase difference π. We observe that there is a bistable region where either the antiphase synchronized state or the desynchronized state can be stabilized. We also find that there exists a hysteresis effect around the dynamic phase transition as the length difference is adiabatically changed.

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The phrase of *odd kind of sympathy* (in short, odd sympathy) was used by a prominent Dutch mathematician and physicist Christiaan Huygens in order to mention an interesting phenomenon observed for the two pendulum clocks attached on the wall [1]: Even after an intentional disturbance, two pendulum clocks evolve into a state in which they swing together in the opposite directions in synchrony. This is one of the most historical observations which naturally lead to the concept of synchronization of dynamic variables, and have attracted many scientific/engineering researchers for a long time [1, 2]. In the language of synchronization study, the odd sympathy corresponds to the antiphase synchronization in which the frequencies of the pendulums' oscillations become identical but the phases show the mismatch π.

There are abundant examples of synchronization phenomena from biological objects [3] and celestial systems [4] in nature to manmade electrical and mechanical systems [5]. Interestingly, such synchronization behaviors have been in most cases empirically discovered by chance *a posteriori* (see the Introduction of Ref. 2), and we still need to understand more to predict why and when it happens. The lack of the precise knowledge can result in a noncontrollable outcome or a catastrophic disaster in the worst case [6]. Even for such a simple setup of two pendulums, which allowed a chance for C. Huygens to become aware of the odd sympathy, investigations have still been performed to unveil the synchronization property in it [2, 7–11]. Such a study on the prototypical setup is necessary for a comprehensive understanding of the synchronization phenomenon as a dynamic phase. This is also to contribute to making the abstract-model-based synchronization research more fruitful in the viewpoint of the synchronization as a thermodynamic phase [13].

More than two hundred years later after the Huygens' observation of the odd sympathy, a qualitative understanding was firstly tried by Korteweg in 1906 [7]. Nearly one hundred more years later, in early 2000s, Bennett *et. al.* revisited this issue [7] and successfully reproduced the antiphase synchronization in their experiment and theoretical model study. Other researchers of the two-pendulum system have also been interested in the inphase synchronization, in which the frequencies of the pendulums' oscillations are identical without the phase mismatch. The possibility of the existence of an inphase state was suggested in Ref. 2, and was credibly reproduced for the first time in the experiment introduced in Ref. 8. These works have been followed by a series of studies where various experimental setups and mechanical models have been proposed to understand the onset of inphase synchronization as well as antiphase one [9–11].

The motivation of this work is to investigate how generic the synchronization is in the two pendulum system. This is attributed to the fact that each work above [7–11] is based on its own specific experimental setup, for example, escapement mechanism and system structure. We are also interested in the bifurcation property of the synchronization state. In this paper, we numerically study the antiphase synchronization of the two nonidentical pendulums in the viewpoint of the dynamic phase transition. For the curiosity of generic feature, we introduce a model whose detail is as simple as possible. As the lengths of pendulums are varied, it is observed that there exists a broad range of bistable region, where the antiphase synchronized state and the desynchronized state coexist depending on initial conditions. The bistability is also shown to lead to a hysteresis effect as the lengths of pendulums are varied adiabatically.

We first introduce the three degrees of freedom model [7], schematically shown in Fig. 1. We apply the Lagrangian least action principle [15] for nonconservative forces, i.e., the escapement and the friction forces in this work, and achieve the equations of motion in dimension-

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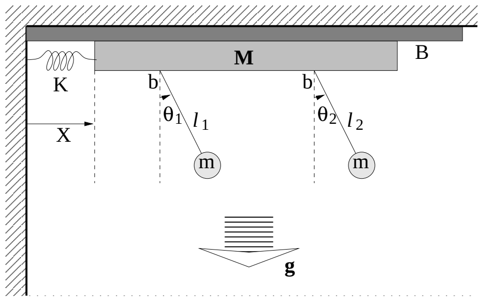

FIG. 1: Each pendulum ($i = 1$ and 2) of the mass m is connected to the rigid common frame of the mass M by the massless rigid rod of the length l_i. The frame at the horizontal position X is attached to a spring with the spring constant K. It is assumed that the sliding motion of the frame and the pivot motion of the pendulum is dissipative, which are described by the friction constant B and b, respectively. The hatched region stands for the rigid immobile wall considered as the reference frame and the uniform gravitational field $\mathbf{g}$ is applied vertically. The three movable objects (the frame and the two pendulums) are described by the coordinates X, θ_1, and θ_2, respectively, which are the three degrees of freedom in the system.

less form (see Fig.1 and compare with Ref. 7)

$$\ddot{\theta}_i + 2\gamma\dot{\theta}_i + (\sin\theta_i + \ddot{x}\cos\theta_i)/\bar{l}_i - f_i = 0, \quad (1)$$

$$\ddot{x} + 2\Gamma\dot{x} + \Omega^2 x + \mu\sum_i \bar{l}_i(\ddot{\theta}_i\cos\theta_i - \dot{\theta}_i^2\sin\theta_i) = 0, \quad (2)$$

where the frame coordinate $x(\equiv X/l)$ and the dimensionless length $\bar{l}_i(\equiv l_i/l)$ are measured in units of l ($\bar{l}_i = 1+\epsilon_i$, where ϵ_i can be interpreted as a small but unavoidable relative error in manufacturing or measurement), and the time t (the dots on the symbols represent time derivatives) is in units of $\sqrt{l/g}$, respectively. We have also defined the reduced mass as $\mu \equiv m/(M+2m)$, the effective coupling strength of the frame as $\Omega^2 \equiv K/(M+2m)$, and the dimensionless friction coefficients for the frame $\Gamma \equiv (B/2)(\sqrt{l/g})/(M+2m)$ and for the pendulum $\gamma \equiv (b/2)\sqrt{l/g}$, respectively. Since every degree of freedom in the system is subject to the damping, the motion will eventually stop at $x = \theta_1 = \theta_2 = 0$ in the absence of the external energy source. As the escapement method, we apply the impulsive force f_i in Eq. (1) when $\dot{\theta}_i = 0$ [$f_i < 0$ for $\theta_i > 0$ and $f_i > 0$ for $\theta_i < 0$]. Our escapement method enforces the pendulums not to stop by injecting kinetic energy into the system. It should be noted that the death phase observed in Ref. 7 is not allowed in our setup. Although other escapement mechanisms could lead to different results, we believe that our escapement algorithm is not too far from the reality, and that the generic qualitative results obtained in this work should be observable in properly prepared real experiments. We also remark that the other mechanical detail such as friction in joints brings about a complex phenomenon, which has been thoroughly studied in Ref. 12)

In the numerical experiments, we use $\gamma = 3\times10^{-4}, \Omega = 0, \Gamma = 0.8$, and $|f_1| = |f_2| = 0.35$. We have tested the reduced mass $\mu = 0.02, 0.025$, and 0.03, only to find insignificant differences, and the results presented in this work are for $\mu = 0.025$. We use the 4th-order Runge-Kutta algorithm to integrate equations of motion with the discrete time step size $\Delta t = 0.01$.

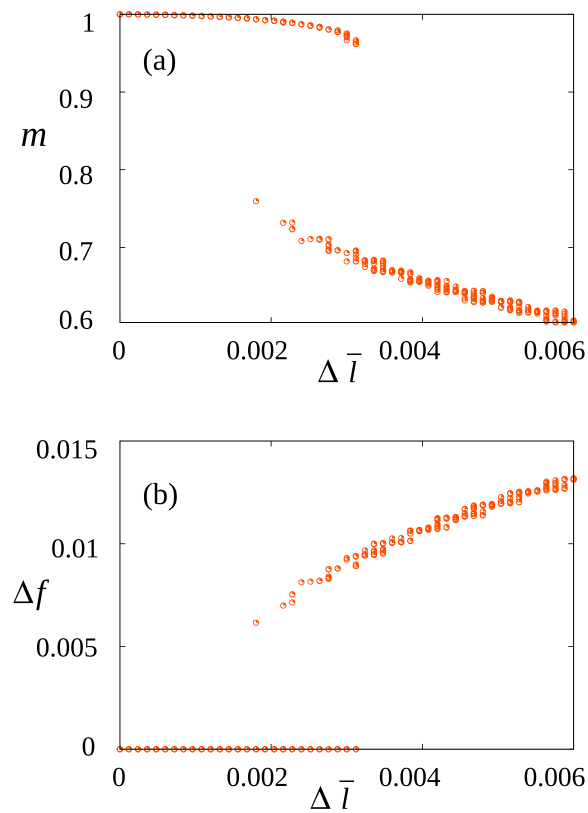

FIG. 2: (Color online) (a) The phase synchronization order parameter m and (b) the frequency entrainment order parameter Δf are shown. The dimensionless length difference $\Delta\bar{l} = |\bar{l}_1 - \bar{l}_2|$ is used for the horizontal axis with $\bar{l}_1 = 1$, and each point is obtained from the random initial condition with $\dot{\theta}_i = 0$ and $\theta_i \in \pm[0.05, 0.1]$. Clearly observed is the existences of two distinct dynamic phases, the antiphase synchronized state and the desynchronized state: The former is characterized by $m \approx 1$ and $\Delta f \approx 0$, while the latter by m significantly less than unity and Δf far from zero. It is to be noted that there is a broad range of bistability where whether the system settles down to the synchronized state or not depends on actual values of $\bar{l}_1$ and $\bar{l}_2$. Change of initial conditions is found to alter neither the dynamic phase transition point nor the observed bistability significantly.

In order to measure the degree of synchrony, we define the order parameter m for the phase synchronization as

$$m \equiv \langle\cos(\Delta\phi - \pi)\rangle, \quad (3)$$

where $\langle\cdots\rangle$ is the time average taken after achieving the steady state and $\Delta\phi \equiv |\phi_1 - \phi_2|$ with the phase ϕ_i determined from $\theta_i \propto \sin\phi_i$. For this, we neglect the result

generated in the initial duration of $T_{\rm ini} = 10^5$, and then take the average during the time $T = 10^4$. Our definition of m gives us $m = 1$ if the two pendulums keep the antiphase synchronization, while $m = -1$ is obtained for the complete inphase synchronization. The smaller $|m|$ is, the worse the synchronization occurs. In addition to the phase synchronization of oscillators, a related but distinct phenomenon is the frequency entrainment [13]. We also gauge the degree of the frequency entrainment simply by measuring the frequency difference defined by

$$\Delta f \equiv \left| \frac{N_1}{T} - \frac{N_2}{T} \right|, \tag{4}$$

where N_i is the number of oscillations of the pendulum i during the time T after the initial transient period $T_{\rm ini}$.

In the numerical computation, we change $\bar{l}_2$ from 1 to 1.006 with the interval 1.2×10^{-4}, for fixed $\bar{l}_1 = 1$ [14]. For each pair of $(\bar{l}_1, \bar{l}_2)$ prepared in this way, we assign the quenched random number $\theta_i \in \pm[0.05, 0.1]$ at the time $t = 0$ and also $x = \dot{x} = \dot{\theta}_1 = \dot{\theta}_2 = 0$ is used as the initial condition. We then measure the two key quantities m and Δf at given values of the length difference

$$\Delta \bar{l} \equiv |\bar{l}_1 - \bar{l}_2|. \tag{5}$$

When $\Delta \bar{l} = 0$, the two pendulums have identical natural frequency and we expect them to show perfect antiphase synchronization [7] to give us $m = 1$ and $\Delta f = 0$. As the length difference becomes larger, it is expected that beyond some value of $\Delta \bar{l}$ the system should stop showing synchronization resulting in $m < 1$ and $\Delta f \neq 0$.

Figure 2 summarizes our main results: As is expected, one can clearly see the existence of dynamic phase transition at $\Delta \bar{l}_c \approx 0.002$ which splits the antiphase synchronized state ($m \approx 1$ and $\Delta f \approx 0$) and the desynchronized state ($m < 1$ and $\Delta f > 0$) as the length difference $\Delta \bar{l}$ is increased. The measured two order parameters are also shown not to have intermediate values around the dynamic phase transition, indicating the discontinuous nature of the transition. Another interesting observation is the existence of the broad range of bistability: Whether or not the system approaches the synchronized state is not uniquely determined by the length difference $\Delta \bar{l}$ only. Although not shown here, we also observe that the use of the larger reduced mass μ increases $\Delta \bar{l}_c$, which has also been reported in Ref. 7.

We next investigate in Fig. 3 the features of the antiphase synchronized state [(a) and (b) for $\Delta \bar{l} = 9.6 \times 10^{-4}$] and the desynchronized state [(c) and (d) for $\Delta \bar{l} = 39.6 \times 10{-4}$] for the reduced mass $\mu = 0.025$. In Fig. 3(a) for the antiphase synchronized state, one finds that there is a manifest anticorrelation between θ_1 and θ_2 due to the phase difference $\Delta\phi = \pi$ after some initial transient period. The approach toward the antiphase synchronization is clearly displayed in Fig. 3(b): Again after transients, $\Delta\phi$ approaches the odd multiple of π. In contrast, the desynchronized state shown in Fig. 3(c) and (d) exhibit very different behaviors: The anticorrelation between θ_1 and θ_2 becomes much weaker [see (c)] and the phase difference $\Delta\phi$ increases indefinitely in time [see (d)], due to the nonzero frequency difference $\Delta f \neq 0$ [see Fig. 2(b)].

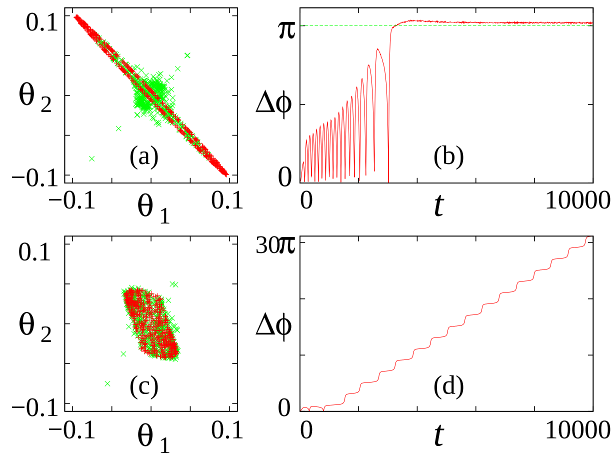

FIG. 3: (Color online) The representative details of pendulum motions are depicted for the antiphase synchronized state [$\Delta \bar{l} = 9.6 \times 10^{-4}$, (a) and (b)] and for the desynchronized state [$\Delta \bar{l} = 39.6 \times 10^{-4}$, (c) and (d)] for $\mu = 0.025$. In (a) and (c), the time evolution of angular coordinates is exhibited in the plane of (θ_1, θ_2), green for $t \leq 5000$ and red for $t > 5000$, and in (b) and (d) the phase difference $\Delta\phi(t)$ is shown as a function of the time t. In the antiphase synchronized state, (a) θ_1 and θ_2 eventually align along the line with the negative slope in the plane (θ_1, θ_2) since (b) $\Delta\phi \to \pi$. The desynchronized state is characterized by (c) the scattered points in the (θ_1, θ_2) plane, and (d) the indefinite increase of $\Delta\phi$ as t is increased due to the mismatch of the frequencies ($\Delta f \neq 0$).

We finally examine that there is a hysteresis effect around the dynamic phase transition between the antiphase synchronized phase and the desynchronized phase. For this, we fix the length of the first pendulum to $\bar{l}_1 = 1$ and change adiabatically $\bar{l}_2$. For the given value of $\bar{l}_2$, we integrate equations of motion for a sufficiently long time to achieve the steady state and change $\bar{l}_2$ by 6.0×10^{-5}. Note that since the hysteresis is of interest now, the system is not reinitialized after the control parameter $\bar{l}_2$ is changed. As clearly shown in Fig. 4, our two pendulum system manifests the hysteresis behavior around the dynamic phase transition: The phase synchronization order parameter m follows different curves when the length of the second pendulum is increased (the red curve in Fig. 4) and decreased (the green one in Fig. 4). All the observations, i.e., the discontinuous nature of the dynamic phase transition, the broad range of bistability, and the strong hysteresis effect, strongly suggest that a subcritical Hopf bifurcation takes place there if stated in the language of nonlinear dynamics [16].

In summary, we have numerically studied the synchronization of the two nonidentical pendulum motions, pivoting on a common movable frame. Within the limitation of our setup of numerical experiments, it has been clearly

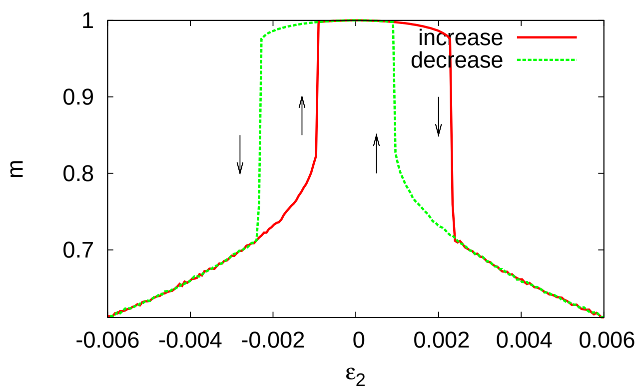

FIG. 4: (Color online) The phase synchronization order parameter m is measured as the length $\bar{l}_2$ of the second pendulum is increased (the full red curve) or decreased (the dotted green curve) adiabatically, while the length of the first pendulum is fixed to $\bar{l}_1 = 1$, and $\bar{l}_2 = 1 + \epsilon_2$. The hysteresis effect is clearly seen around the dynamic phase transition splitting the antiphase synchronized (the upper branch of the curves) state and the desynchronized state (the lower branch).

shown that the system exhibits the discontinuous dynamic phase transition from the antiphase synchronized state to the desynchronized state as the length difference is increased. We have also shown that the discontinuous nature of the transition is reflected to the broad range of bistability and also to the existence of the hysteresis effect. We believe that our model reproduces the odd sympathy C. Huygens observed in the seventeenth century. We finally remark that it is well-known that the dynamic property of nonlinear system is significantly affected by such mechanical details if involved in the nonlinearity [17]. In the present system, it is the escapement mechanisms [7, 8] or the structural design of the system [10, 11], for example. Therefore, it is still necessary to examine the various nonlinearity setups for the sound understanding of the synchronization phenomenon in general as a dynamic phase in statistical physics.

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[14] We consider $\Delta\bar{l}$ not only to examine when the synchronization begins but also to reflect the generic inhomogeneity of the real pendulums. The main interest of synchronization phenomena is indeed on the transition point where the dynamic process overcomes the intrinsic difference, arriving at the sympathy. In the simple setup prsented here, the difference of the natural frequencies of the two pendulums is surely the first candidate to prohibit them from synchronizing to each other. We consider $\Delta\bar{l}$ to effectively introduce the natural frequency difference, playing the role of the variance in the frequency distribution in the Kuramoto model (see Ref. 13 and references therein for example).

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